On the non-linear mechanics of wave disturbances in stable and unstable parallel flows

Part 1. The basic behaviour in plane Poiseuille flow

By J. T. STUART
National Physical Laboratory, Teddington, Middlesex

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This paper considers the nature of a non-linear, two-dimensional solution of the Navier–Stokes equations when the rate of amplification of the disturbance, at a given wave-number and Reynolds number, is sufficiently small. Two types of problem arise: (i) to follow the growth of an unstable, infinitesimal disturbance (supercritical problem), possibly to a state of stable equilibrium; (ii) for values of the wave-number and Reynolds number for which no unstable infinitesimal disturbance exists, to follow the decay of a finite disturbance from a possible state of unstable equilibrium down to zero amplitude (subcritical problem). In case (ii) the existence of a state of unstable equilibrium implies the existence of unstable disturbances. Numerical calculations, which are not yet completed, are required to determine which of the two possible behaviours arises in plane Poiseuille flow, in a given range of wave-number and Reynolds number.

It is suggested that the method of this paper (and of the generalization described by Part 2 by J. Watson) is valid for a wide range of Reynolds numbers and wave-numbers inside and outside the curve of neutral stability.

1. Introduction

The present paper and the paper by Watson (1960), forming Part 2 of this study, contain some new developments in the non-linear theory of the mechanics of instability, and may be regarded as following earlier work on the subject by Landau (1944), Meksyn & Stuart (1951), Gorkov (1957), Malkus & Veronis (1958), Stuart (1956*a*, *b*, 1958), Veronis (1959) and Stuart & Watson (1960). For detailed discussions of some of the non-linear effects in instability, the reader is referred to a paper by Stuart (1958); however, it is necessary here to describe some of the results which were obtained earlier and to explain their relationship to the analysis described in the present paper and in Part 2.

Of the papers mentioned above, those of Gorkov (1957), Malkus & Veronis (1958), Stuart & Watson (1960) and Veronis (1959) are concerned with the problem of thermal-convective instability when a horizontal layer of fluid is heated from below. For a description of this kind of instability the reader is referred to the article by Malkus & Veronis (1958). The essential features are the following; when a horizontal layer of fluid at rest is heated from below, it has a tendency towards instability because the hotter fluid is less dense and therefore tends to convect upwards. However, convection only takes place if this thermally induced buoyancy can overcome the twin effects of thermal diffusion and viscous retardation; this happens when a certain parameter, the Rayleigh number ($\mathscr{R}$), exceeds a critical value ($\mathscr{R}_c$). (Mathematically, this is known as the phenomenon of 'branching' of the solution of a differential equation, at a given value of a parameter.) Then the fluid convects into cells of polygonal planform—that is, the flow is periodic in both of the horizontal directions. According to linearized theory, the velocity increases exponentially with time until it becomes too large for the validity of the linearized theory. On the other hand, observations show that, for $\mathscr{R} > \mathscr{R}_c$, a steady, equilibrium state of convection exists. A theory in accordance with the latter evidence has been developed independently by Gorkov (1957) and by Malkus & Veronis (1958). For a given wave-number, they obtained the solution of the non-linear equations as an amplitude perturbation about the (neutral) solution (of linearized theory), which is valid at the critical Rayleigh number. This procedure yields a solution giving a possible equilibrium state of finite-amplitude convection. However, they did not show that the amplification of the unstable solution of linearized theory, at a given Rayleigh number, does lead to this steady equilibrium state. The latter is a problem of showing that the instability of one equilibrium state (namely the original layer of fluid at a Rayleigh number above the critical) of the fluid leads to another equilibrium state (namely that of steady finite-amplitude convection). This aspect of the problem has been studied by Stuart & Watson (1960), who have shown that the unstable solutions of linearized theory do grow in amplitude until they approach the equilibrium state, at least for Rayleigh numbers close enough to the critical.

The papers of Gorkov and Malkus & Veronis on the one hand, and that of Stuart & Watson (1960) on the other hand, illustrate two different approaches to the non-linear problem arising from hydrodynamic instability. In the first approach, the change of the equilibrium state is followed as the Rayleigh number is raised; the question of growth or decay of a disturbance with time is not studied. In the second approach, the Rayleigh number is fixed and the development of the solution with time is followed, until an equilibrium state (if one exists) is reached. Whichever approach to the problem is used, the equilibrium state at a given Rayleigh number is presumably the same.

It should be mentioned here that, in connexion with the second of these approaches, Landau (1944) conjectured that the square of the amplitude ($|A|^2$) of a finite disturbance will behave like the solution of the equation

$$\frac{d|A|^2}{dt} = k_1|A|^2 + k_2|A|^4,$$

where t is the time and k_1 and k_2 are constants. This equation gives an approximation to the amplitude behaviour discussed in the paper by Stuart & Watson, and is the equation also derived by Stuart (1958) from an energy principle.

In the present paper it is intended to study the non-linear problem arising from instability in plane Poiseuille flow by means of the second of the two approaches mentioned above. We consider, therefore, the growth or decay of a disturbance in Poiseuille flow at a fixed Reynolds number, the latter being the appropriate parameter in this problem. The problem of linearized instability in plane Poiseuille flow may be said to be well understood (see, for example, Lin 1955). The minimum critical Reynolds number, $U_0 h/\nu$, is about 5780, where U_0 is the maximum speed in the flow, h is half the distance between the two planes and ν is the kinematic viscosity. The dimensionless wave-number α at which the minimum critical Reynolds number occurs is about 1·02. These calculated results were obtained by Thomas (1953) on a digital computer.

The non-linear problem of instability, when the linearized problem is governed by the so-called Orr–Sommerfeld equation (that is, when the disturbance is a travelling wave), has been considered by Noether (1921), Heisenberg (1924), Meksyn & Stuart (1951) and Stuart (1958). Noether's paper is concerned primarily with plane Couette flow, and gives equations for the case in which non-linearity is included only to the extent of the Reynolds-stress effect in the equation of mean motion; the terms representing the generation of harmonics of the basic disturbance are ignored in the analysis. Heisenberg's paper studies similar equations to those of Noether, both for plane Couette flow and for plane Poiseuille flow. Neither of the papers mentioned gives solution of the equations mentioned: however, Meksyn & Stuart gave an approximate method of solving the non-linear equations of Noether and Heisenberg for plane Poiseuille flow, and used it to obtain an approximate relation between the critical Reynolds number and the amplitude of the disturbance. This relation shows that, as the amplitude of the disturbance rises, the critical Reynolds number for instability drops. It follows that there may be finite-amplitude solutions of the equations of motion at Reynolds numbers and wave-numbers for which the flow is stable according to linearized theory. We shall refer to flows which exhibit this feature as being 'subcritical', or as permitting disturbances under subcritical conditions.

On the other hand, Stuart (1958) has performed a different calculation for plane Poiseuille flow, using the energy-balance equation for the disturbance. At a given Reynolds number, the disturbance velocity was assumed to have a shape (in the space dimensions) given by linearized theory, while at the same time having as amplitude an unspecified function of time. The energy-balance equation then yielded an equation for the amplitude which showed that, at Reynolds numbers above the critical for a given wave-number, an unstable disturbance amplifies until it reaches an equilibrium amplitude. We may refer to flows which exhibit this feature as being 'supercritical', or as permitting disturbances under supercritical conditions.

Clearly the result obtained by Stuart (1958) is different from that obtained by Meksyn & Stuart (1951). It is desirable to know whether it is possible for both phenomena to exist in the same basic flow or whether one of the results mentioned above is a consequence of the approximations made. It was partly to resolve this question that the work described in the present paper was contemplated. It is our aim here to consider the nature of the limiting non-linear solution of the Navier–Stokes equations when the Reynolds number tends to a critical value, namely a value for which a disturbance is neutrally stable. A detailed analysis is given which leads to a formidable numerical problem of the solution of some ordinary differential equations. The essential result of the analysis, barring a particular unlikely contingency, is to show that equilibrium disturbances exist either under subcritical conditions or under supercritical conditions near to a particular critical Reynolds number. The numerical analysis, to which reference has been made, will, when completed, show which is the case. In the event of the unlikely contingency mentioned above, the problem would be both subcritical and supercritical near to a given Reynolds number. This would mean, physically, that non-linear, finite-amplitude equilibrium flows could exist both above and below the critical Reynolds number. Furthermore, this unlikely contingency would mean that the present limiting solution would not be a valid approximation to Watson's (1960) expansions mentioned below.

It is to be emphasized that the present paper is concerned only with the most important terms in the non-linear solution when the Reynolds number tends to a critical value. The related, and important, problem of the development of a valid perturbation expansion of the non-linear, time-dependent Navier–Stokes equations has been solved by Watson (1960) in an accompanying paper. The reader will probably be interested to know of the relevance of the remarks of Lin (1958) to the work described in this paper and that of Watson (1960). In his paper, Lin made the following statement, which summarized his conclusion from some mathematical analysis: 'One of the purposes of this paper is to bring out the remarkable fact, that *for disturbances in a parallel flow, all the harmonic components simultaneously become important around the critical layer, before the amplitude of the fundamental component is large enough to cause any significant distortion of the mean flow.*' This statement will be seen to be at variance with the results of the present paper, where the basic perturbation is of small order A (say), the first harmonic of this and the distortion of the mean motion are of order A^2, and higher harmonic components are of order A^n $(n \geqslant 3)$; moreover, in comparison with linearized theory, our analysis does produce a significant change of the character of the solution, namely the possibility of equilibrium states of finite amplitude, without the generation of a large number of harmonic components. Therefore Lin's statement quoted above is invalid because his analysis does not encompass all possible disturbances; for example, as we shall now show, it ignores some non-linear disturbances of smaller magnitude, namely these discussed in this paper.

In the present work and that of Watson (1960) the non-linear terms become significant when they are of the same order of magnitude (A^3) as the *whole group* of linearized terms. (The largest of the latter terms are of order A, and the result of setting equal to zero the sum of such terms is the Orr–Sommerfeld equation. The remaining terms have order A^3. For an analogy we may note that, although A and $(A+A^3)$ are both of order A, their difference is A^3.) On the other hand, Lin studies disturbances such that a typical non-linear term is of the same order of magnitude as *some* linear term; his analysis therefore considers much larger disturbances than those of this paper and Watson's and implies that Fourier analysis is not applicable to such amplitudes. For a detailed discussion of these matters the reader is referred to §§ 3 and 4 of this paper.

2. Basic equations

Let us consider Poiseuille flow under pressure between two parallel planes, which are set at a distance $2h$ apart. In laminar undisturbed flow, a uniform pressure gradient produces a velocity distribution which is independent of x and has its maximum value U_0 at the centre of the channel. In the following analysis, the reference length is h, the reference velocity U_0 and the reference time h/U_0. We let x denote the co-ordinate parallel to the planes and z the co-ordinate normal to them. The corresponding velocity components are u and w, ψ is the stream function and t is the time.

The governing differential equation of the two-dimensional motion is

$$\frac{\partial \zeta}{\partial t}+\frac{\partial \psi}{\partial z}\frac{\partial \zeta}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial z} = \frac{1}{R}\nabla^2\zeta, \tag{2.1}$$

where

$$\zeta = -\nabla^2\psi, \tag{2.2}$$

$R = U_0h/\nu$ is the Reynolds number and ν denotes the kinematic viscosity. In undisturbed laminar flow, the motion is parallel to the planes and is given by

$$\overline{u}_l = \frac{\partial \psi_l}{\partial z} = 1-z^2. \tag{2.3}$$

A bar above a symbol denotes a mean with respect to x, while the suffix l is used to denote the case of undisturbed laminar flow.

Our object is to examine the stability of the velocity profile (2.3) with respect to a disturbance in the form of a two-dimensional travelling wave. We therefore assume

$$\begin{aligned}\psi = \phi_0(z,t) &+ \phi_1(z,t)\exp[i\alpha(x-c_rt)]+\tilde{\phi}_1(z,t)\exp[-i\alpha(x-c_rt)]\\ &+\phi_2(z,t)\exp[2i\alpha(x-c_rt)]+\tilde{\phi}_2\exp[-2i\alpha(x-c_rt)]\\ &+\ldots,\end{aligned} \tag{2.4}$$

where the symbol $\sim$ denotes a complex conjugate. The quantity α is the (positive) wave-number and c_r is the wave velocity of linearized theory for given α and R. The functions ϕ depend on t to account for any growth or decay of the disturbance; furthermore, any variation of wave velocity with amplitude is also accounted for by the dependence of ϕ_1, etc., on t. The mean velocity, $\overline{u} = \partial\phi_0/\partial z$, is different from that of laminar flow (2.3) because of interaction between the mean flow and the disturbance (see Stuart 1958, p. 3).

Substituting (2.4) into (2.1) and separating out the harmonic components, we obtain

$$\begin{aligned}\left(\overline{u}-c_r-\frac{i}{\alpha}\frac{\partial}{\partial t}\right)(\phi_1''-\alpha^2\phi_1)-\overline{u}''\phi_1+\frac{i}{\alpha R}(\phi_1^{\mathrm{IV}}-2\alpha^2\phi_1''+\alpha^4\phi_1)\\ = \phi_2'(\tilde{\phi}_1''-\alpha^2\tilde{\phi}_1)+2\phi_2(\tilde{\phi}_1'''-\alpha^2\tilde{\phi}_1')-2\tilde{\phi}_1'(\phi_2''-4\alpha^2\phi_2')\\ -\tilde{\phi}_1(\phi_2'''-4\alpha^2\phi_2')+O(\tilde{\phi}_2\phi_3),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\left(\overline{u}-c_r-\frac{i}{2\alpha}\frac{\partial}{\partial t}\right)(\phi_2''-4\alpha^2\phi_2)-\overline{u}''\phi_2+\frac{i}{2\alpha R}(\phi_2^{\mathrm{IV}}-8\alpha^2\phi_2''+16\alpha^4\phi_2)\\ = -\tfrac{1}{2}(\phi_1'\phi_1''-\phi_1\phi_1''')+O(\tilde{\phi}_1\phi_3),\end{aligned} \tag{2.6}$$

$$\frac{\partial^2\overline{u}}{\partial t\,\partial z}+i\alpha\frac{\partial^2}{\partial z^2}\{(\phi_1'\tilde{\phi}_1-\tilde{\phi}_1'\phi_1)+2(\phi_2'\tilde{\phi}_2-\tilde{\phi}_2'\phi_2)+\ldots\} = \frac{1}{R}\frac{\partial^3\overline{u}}{\partial z^3}, \tag{2.7}$$

together with equations conjugate to (2.5) and (2.6) and equations involving ϕ_3, ϕ_4, etc. Primes denote derivatives with respect to z. Thus, we have an infinite set of differential equations for an infinite set of dependent variables, $\bar{u}$, ϕ_1, ϕ_2, etc.

Equation (2.7) may be integrated once to give

$$\frac{\partial \bar{u}}{\partial t}+i\alpha\frac{\partial}{\partial z}\{\phi_1'\tilde{\phi}_1-\tilde{\phi}_1'\phi_1+2(\phi_2'\tilde{\phi}_2-\tilde{\phi}_2'\phi_2)+\dots\}=\frac{2}{R}+\frac{1}{R}\frac{\partial^2\bar{u}}{\partial z^2}, \tag{2.8}$$

where the arbitrary function of time (the pressure gradient) has been chosen to have the constant value $(2/R)$, which is the value of the pressure gradient in the undisturbed laminar flow. A boundary condition on the motion, there, is that the externally applied pressure gradient is to remain unchanged, despite any growth or decay of the disturbance. (Another possible boundary condition is that of constant mass flux; see Watson (1960).)

The boundary conditions on the functions $\bar{u}$ and ϕ_n are that

$$\bar{u}=\phi_n=\phi_n'=0 \quad \text{at} \quad z=\pm 1, \tag{2.9}$$

because the velocity must be zero at a solid boundary. Since $\bar{u}$ is to be an even function of z, it can be seen from equations (2.5) and (2.6) that ϕ_1 may be either even or odd, with ϕ_2 correspondingly odd or even. Equation (2.8), and higher-order equations for ϕ_n, are consistent with this. We shall consider the case of ϕ_1 even, and then the boundary conditions on $\bar{u}$ and ϕ_n are

$$\left.\begin{aligned}\bar{u}=\phi_n=\phi_n'=0 \quad \text{at} \quad z=+1\\ \bar{u}'=\phi_{2n-1}'=\phi_{2n-1}'''=\phi_{2n}=\phi_{2n}''=0 \quad \text{at} \quad z=0\end{aligned}\right\} \quad (n=1,2,3,\dots). \tag{2.10}$$

It is necessary also to specify a condition on the solution of the equations with regard to the time dependence. Such a condition may conveniently be referred to as an 'initial' condition, though we shall find, in some cases, that the condition has to be applied as $t\to+\infty$, and is thus a 'terminal' condition rather than an initial condition.

Let us consider first the case of a disturbance under supercritical conditions (Stuart 1958), that is, one which amplifies for small amplitudes. Our object in such a case is to calculate the development of the disturbance with the passage of time. A suitable initial condition, therefore, is that the function ϕ_1 shall be an exponentially increasing function of t in the limit as $t\to-\infty$; in fact, ϕ_1 has to be the appropriate function, $\psi_1(z)\exp(\alpha c_i t)$, where $c_i>0$, of the linearized-instability theory.

On the other hand, the situation is somewhat different when we consider a disturbance under subcritical conditions; in this case, a small disturbance does not amplify, but is damped. There is no question of applying an 'initial' condition, in the sense that it is used in the case of disturbances under supercritical conditions. However, we can apply a 'terminal' condition, namely that the function ϕ_1 shall be an exponentially decreasing function of t in the limit as $t\to+\infty$; by analogy with the supercritical case, ϕ_1 has to be the function $\psi_1(z)\exp(\alpha c_i t)$, where $c_i<0$, of the linearized stability theory. The problem is to study the non-linear behaviour, at finite values of t, of a disturbance which decays to zero, as $t \to \infty$, through the linearized-stability solution.

The subcritical problem with a 'terminal' condition may, from the present point of view, be regarded as the inverse of the supercritical problem with an initial condition. In either case, the initial or terminal conditions on the functions ϕ_m $(m > 1)$ and $\bar{u}$ follow automatically, when the condition on ϕ_1 is specified, if the solution of equations (2.5), (2.6), etc., is expanded in powers of $\exp(\alpha c_i t)$. Such an expansion is presumably valid for small amplitudes, that is for

$$\text{(i) } c_i > 0,\ t \to -\infty, \quad \text{and} \quad \text{(ii) } c_i < 0,\ t \to +\infty.$$

Linearized theory

The linearized theory of instability is based on the neglect of all terms which are quadratic in the functions ϕ_1, ϕ_2, etc. Then, because of linearity, ϕ_1 may be assumed to be proportional to $\exp(\alpha c_i t)$. If we write

$$\phi_1(z,t) = C\exp(\alpha c_i t)\,\psi_1(z), \tag{2.11}$$

equation (2.5) reduces to the Orr–Summerfeld equation

$$L\psi_1 \equiv (\bar{u}_l - c_r - ic_i)(\psi_1'' - \alpha^2\psi_1) - \bar{u}_l''\psi_1 + \frac{i}{\alpha R}(\psi_1^{\mathrm{IV}} - 2\alpha^2\psi_1'' + \alpha^4\psi_1) = 0. \tag{2.12}$$

The appropriate solution of equation (2.8) is the steady, laminar flow $\bar{u}_l = 1 - z^2$ because, with the Reynolds stress neglected, there is no reason for $\bar{u}$ to vary with time ($\partial\bar{u}/\partial t = 0$). The boundary conditions for the solution of (2.12), namely

$$\left.\begin{aligned} \psi_1 = \psi_1' &= 0 \quad \text{at} \quad z = +1, \\ \psi_1' = \psi_1''' &= 0 \quad \text{at} \quad z = 0, \end{aligned}\right\} \tag{2.13}$$

define an eigenvalue problem for α, R, c_r and c_i. The (complex) eigenrelation yields two real relationships between these four quantities. If c_i is specified, α and R (say) are each known as functions of c_r. For the case $c_i = 0$, a plot of α against R yields a curve of 'neutral stabilty' (see figure 1). Within this curve c_i is positive and the disturbance amplifies; the flow is therefore unstable. Outside the neutral curve, c_i is negative and the flow is stable.

For full details of the theory, the reader is referred to the book by Lin (1955), and to the reference papers given there. A paper of particular interest for our present purposes is that of Thomas (1953), who solved (2.12) on a digital computing machine and obtained several sets of eigenvalues. The critical Reynolds number was found by interpolation to be $R_c = 5780$ at $\alpha = 1{\cdot}02$. Furthermore, Thomas calculated the eigenfunction, $\psi_1(z)$, for the case

$$\alpha = 1, \quad R = 10^4, \quad c_r = 0{\cdot}2375, \quad c_i = 0{\cdot}0037$$

(see figure 1 of Stuart 1958).

In the present paper we shall refer to the inside of the neutral curve described above as the supercritical region, because the disturbances there amplify for small amplitudes. The region outside the neutral curve (particularly, the region to the left of the neutral curve) will be referred to as the subcritical region, because the disturbances there decay for small amplitudes.

A possible way of obtaining a formal solution of equations (2.5), (2.6), (2.8), etc., and thereby of generalizing the solution of linearized theory, is to expand the solution in powers of $\exp(\alpha c_i t)$. This expansion is likely to be valid provided the latter function is small; in the supercritical case, $c_i > 0$, this implies $t \to -\infty$; whereas, in the subcritical case, $c_i < 0$, it implies $t \to +\infty$.

It is readily seen that such an expansion of the solution would lead to series of the form

$$\left.\begin{aligned} \phi_1(z,t) &= \psi_{11}(z)\exp(\alpha c_i t) + \psi_{13}(z)\exp(3\alpha c_i t) + \ldots, \\ \phi_2(z,t) &= \psi_{22}(z)\exp(2\alpha c_i t) + \psi_{24}(z)\exp(4\alpha c_i t) + \ldots, \end{aligned}\right\} \tag{2.14}$$

and it is abundantly clear that the series diverge when $\exp(\alpha c_i t)$ is large. In the supercritical case, $c_i > 0$, this happens when $t \to +\infty$ whereas, in the subcritical case, $c_i < 0$, it happens when $t \to -\infty$.

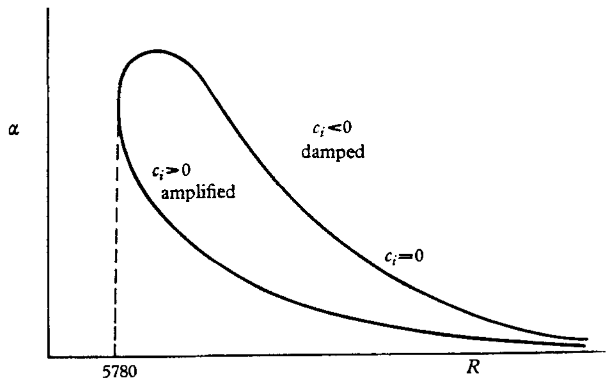

FIGURE 1. Neutral curve.

It may be said that a principal object of the present paper, and that of Watson (1960), is to devise a form of expansion which is *valid at all times* and which yields an equilibrium state, if one exists. Such an expansion involves, formally, a rearrangement of the terms in the series (2.14). The present paper shows how the leading, dominant, terms of such a new expansion may be obtained, while the paper by Watson (1960) develops a rigorous expansion of the solution of the Navier–Stokes equations.

3. A simplification of the non-linear problem of instability in the limit of small amplification or damping

A glance at equations (2.5), (2.6), (2.8) and the corresponding equations for the higher harmonics shows that an attempt to obtain a solution of the general problem, involving an infinite set of differential equations, would present considerable difficulties. For this reason, it seems worth examining whether there is some limiting state, as a characteristic Reynolds number is approached, for which the infinite set of differential equations may be reduced to a finite set.

In a recent paper (Stuart 1958) it is suggested that there may be an equilibrium solution of the equations, the square of whose amplitude is proportional to the difference between the actual and critical Reynolds numbers, provided this difference is small compared with the critical Reynolds number. We shall therefore examine (2.5), (2.6) and (2.8) on the assumption that the amplitude (of ϕ_1) is never of greater order of magnitude than $(R-R_c)^{\frac{1}{2}}$, where R_c is the critical Reynolds number (for a given wave-number, α). It is more convenient to regard ϕ_1 as of order $c_i^{\frac{1}{2}}$. (It is known from linearized theory that c_i is proportion to $R-R_c$.) We also assume that $\partial/\partial t$ is never of greater magnitude than c_i, which is its order of magnitude according to linearized theory.

Consider first equation (2.6); since ϕ_1 is of order $c_i^{\frac{1}{2}}$, it can be seen that ϕ_2 is of order c_i. (We return later to a discussion of the terms of order $\tilde{\phi}_1\phi_3$.) Moreover, it can be seen from the related equation for the function ϕ_3 that ϕ_3 is of order $c_i^{\frac{3}{2}}$. From the mean motion equation (2.8), it can be seen that the distortion of the mean flow ($F = \bar{u}-\bar{u}_l$) is of order c_i; this arises from that part of the Reynolds stress which involves products of ϕ_1. Thus we have the following orders of magnitude:

$$\phi_1 \sim c_i^{\frac{1}{2}}; \quad \phi_2 \sim c_i; \quad \phi_3 \sim c_i^{\frac{3}{2}}; \quad F = \bar{u}-\bar{u}_l \sim c_i; \quad \partial/\partial t \sim c_i. \tag{3.1}$$

Now consider the higher-order terms in equations (2.5), (2.6) and (2.8) when $c_i \to 0$. In equation (2.5) the terms $O(\tilde{\phi}_2\phi_3)$ are of order $c_i^{\frac{5}{2}}$, and so can be neglected compared with the dominant terms $(\tilde{\phi}_1\phi_2)$ on the right-hand side, since the latter are of order $c_i^{\frac{3}{2}}$. Although the (linear) terms on the left-hand side of equation (2.5) appear, at first sight, to be of order $c_i^{\frac{1}{2}}$, we shall show shortly that they are of order $c_i^{\frac{3}{2}}$, and therefore are balanced by the dominant terms on the right-hand side of (2.5). In equation (2.6) the terms $O(\tilde{\phi}_1\phi_3)$ are of order c_i^2, and can be neglected because they are of higher order than the dominant terms (of order c_i) on the right-hand side. Furthermore, in equation (2.8) the terms $\partial\bar{u}/\partial t$ and the Reynolds stress-term involving the square of ϕ_2 are both small (of order c_i^2) compared with the dominant Reynolds stress-term. A most important result of the above estimate of orders of magnitude is that the functions ϕ_3, ϕ_4, etc., do not affect the three equations (2.5), (2.6) and (2.8) in the limit as $c_i \to 0$. We may therefore terminate the Fourier series after the ϕ_2 term, and the equations governing the problem become

$$\begin{aligned}\left(\bar{u}-c_r-\frac{i}{\alpha}\frac{\partial}{\partial t}\right)(\phi_1''-\alpha^2\phi_1)-\bar{u}''\phi_1+\frac{i}{\alpha R}(\phi_1^{\mathrm{IV}}-2\alpha^2\phi_1''+\alpha^4\phi_1)&\\ =\phi_2'(\tilde{\phi}_1''-\alpha^2\tilde{\phi}_1)+2\phi_2(\tilde{\phi}_1'''-\alpha^2\tilde{\phi}_1')-2\tilde{\phi}_1'(\phi_2''-4\alpha^2\phi_2)-\tilde{\phi}_1(\phi_2'''-4\alpha^2\phi_2'),&\end{aligned} \tag{3.2}$$

$$\begin{aligned}(\bar{u}_l-c_r)(\phi_2''-4\alpha^2\phi_2)-\bar{u}_l''\phi_2+\frac{i}{2\alpha R}(\phi_2^{\mathrm{IV}}-8\alpha^2\phi_2''+16\alpha^4\phi_2)&\\ =-\tfrac{1}{2}(\phi_1'\phi_1''-\phi_1\phi_1'''),&\end{aligned} \tag{3.3}$$

$$i\alpha\frac{\partial}{\partial z}\{\phi_1'\tilde{\phi}_1-\tilde{\phi}_1'\phi_1\}=\frac{2}{R}+\frac{1}{R}\frac{\partial^2\bar{u}}{\partial z^2}. \tag{3.4}$$

These three equations, together with the complex conjugates of (3.2) and (3.3), form a set of five equations for the five functions ϕ_1, $\tilde{\phi}_1$, ϕ_2, $\tilde{\phi}_2$ and $\bar{u}$.

In equation (3.3) the terms proportional to $\bar{u}-\bar{u}_l$, to c_i and to $\partial/\partial t$ have been omitted because such terms are of order c_i^2 and therefore of higher order than the terms retained. We turn now to a study of equation (3.2). According to the linearized instability theory the sum of the (linearized) terms on the left-hand side of the equation is equal to zero. Within the framework of the present theory, this condition is replaced by the feature that the set of linearized terms is balanced by a set of non-linear terms which are of smaller order ($c_i^{\frac{3}{2}}$) than ϕ_1, which is of order $c_i^{\frac{1}{2}}$. This apparent disparity in orders of magnitude can be resolved in the following way. If we seek a solution of the form

$$\phi_1 = A_1(t)\,\psi_1(z) + O(c_i^{\frac{3}{2}}), \tag{3.5}$$

where A_1 is of order $c_i^{\frac{1}{2}}$, then we wish to ensure the cancellation of terms of order $c_i^{\frac{1}{2}}$. This can be achieved by choosing $\psi_1(z)$ to be the normalized stream function of linearized theory, and then the next higher-order terms in (3.2) are of magnitude $c_i^{\frac{3}{2}}$. A method of calculating the term of order $c_i^{\frac{3}{2}}$ in (3.5) is given in § 4.

Before proceeding to further study of the solution of equations (3.2), (3.3) and (3.4), it is desirable to discuss the restrictions on the magnitude of c_i which are necessary if our approximations are to be valid. In this connexion, it will be noticed that because $(\alpha R)^{-1}$ is very small (of order 10^{-4}) and multiplies the highest derivative of equation (2.12) the solution of that equation has a 'near-singularity' at the point where $\overline{u} = c_r + ic_i$; this is the so-called 'critical' layer (Lin 1955). It is known from the linearized instability theory that this layer of rapid change of velocity has a thickness of order $\epsilon \doteq (\alpha R)^{-\frac{1}{3}}$. The existence of the critical layer is naturally relevant to the way in which the factor

$$\overline{u} - c_r - (i/\alpha)\,\partial/\partial t$$

is treated in the non-linear analysis leading to equations (3.2) and (3.3). It seems likely that any change (of order c_i) in the position of the critical point, due to change of $\overline{u}$ and c_r, must be very small compared with the magnitude of the thickness of the critical layer. This suggests that the Fourier expansion (2.4), of which (3.2)–(3.4) forms the basic approximation, will converge provided

$$c_i \ll (\alpha R)^{-\frac{1}{3}}. \tag{3.6}$$

If this condition is satisfied the treatment of the left-hand side of (3.2) and the neglect of $\overline{u} - \overline{u}_l$ and $\partial/\partial t$ in equation (3.3) are likely to be valid. A discussion of the range of Reynolds numbers and wave-numbers for which (3.6) is valid is given in § 6.

4. A method of solution of the basic non-linear problem in the limit $c_i \to 0$

We look for a solution of equations (3.2), (3.3) and (3.4) in the form

$$\left.\begin{aligned} \phi_1 &= A_1(t)\,\psi_1(z) + A_{11}(t)\,\psi_{11}(z), \\ \phi_2 &= A_1^2(t)\,\psi_2(z), \\ \overline{u} &= \overline{u}_l + A_1\tilde{A}_1 f(z) = 1 - z^2 + A_1\tilde{A}_1 f(z), \end{aligned}\right\} \tag{4.1}$$

where A_1 is of order $c_i^{\frac{1}{2}}$ and A_{11} is of order $c_i^{\frac{3}{2}}$. Although in the limit $c_i \to 0$, only the first term of ϕ_1 is important, both are of importance for the solution of the differential equation (3.2). This is because the term $\psi_1(z)$ is the eigenfunction of linearized theory for the given α and R, so that the order of magnitude of terms on the left-hand side of (3.2) arising from $A_1\psi_1$ is only $c_i^{\frac{3}{2}}$, terms of order $c_i^{\frac{1}{2}}$ having cancelled; on the other hand, the term $A_{11}\psi_{11}$ must be retained on the left-hand side of (3.2) because its order of magnitude is already $c_i^{\frac{3}{2}}$, and is not altered because ψ_{11} is *not* an eigenfunction. The assumption (4.1) has been generalized by Watson (1960) into a valid expansion.

Substituting (4.1) into (3.2), (3.3) and (3.4), and using (2.12), we obtain

$$\left(ic_i A_1 - \frac{i}{\alpha}\frac{dA_1}{dt}\right)(\psi_1'' - \alpha^2\psi_1)$$
$$+ A_{11}\left[(\bar{u}_l - c_r)(\psi_{11}'' - \alpha^2\psi_{11}) - \bar{u}_l''\psi_{11} + \frac{i}{\alpha R}(\psi_{11}^{\mathrm{IV}} - 2\alpha^2\psi_{11}'' + \alpha^4\psi_{11})\right]$$
$$= A_1^2\tilde{A}_1 g(z), \tag{4.2}$$

$$(\bar{u}_l - c_r)(\psi_2'' - 4\alpha^2\psi_2) - \bar{u}_l''\psi_2 + \frac{i}{2\alpha R}(\psi_2^{\mathrm{IV}} - 8\alpha^2\psi_2'' + 16\alpha^4\psi_2) = -\tfrac{1}{2}(\psi_1'\psi_1'' - \psi_1\psi_1'''), \tag{4.3}$$

$$i\alpha\frac{d}{dz}\{\psi_1'\tilde{\psi}_1 - \tilde{\psi}_1'\psi_1\} = \frac{1}{R}\frac{d^2f}{dz^2}. \tag{4.4}$$

The function $g(z)$ in (4.2) is defined by

$$g(z) \equiv \psi_2'(\tilde{\psi}_1'' - \alpha^2\tilde{\psi}_1) + 2\psi_2(\tilde{\psi}_1''' - \alpha^2\tilde{\psi}_1')$$
$$- 2\tilde{\psi}_1'(\psi_2'' - 4\alpha^2\psi_2) - \tilde{\psi}_1(\psi_2''' - 4\alpha^2\psi_2')$$
$$- f(\psi_1'' - \alpha^2\psi_1) + f''\psi_1. \tag{4.5}$$

We also write
$$g_1(z) \equiv \psi_1'' - \alpha^2\psi_1. \tag{4.6}$$

It will be noticed that the function ψ_{11} is of importance only in certain terms on the left-hand side of (3.2), and can be ignored on the right-hand side of (3.2) and in (3.3) and (3.4), because there it yields terms of higher order. The boundary conditions may be obtained from (2.10) and are

$$\left.\begin{aligned} f = \psi_1 = \psi_1' = \psi_{11} = \psi_{11}' = \psi_2 = \psi_2' = 0 \quad &\text{at} \quad z = 1,\\ f' = \psi_1' = \psi_1''' = \psi_{11}' = \psi_{11}''' = \psi_2 = \psi_2'' = 0 \quad &\text{at} \quad z = 0. \end{aligned}\right\} \tag{4.7}$$

Once the eigenvalue problem (2.12) has been solved for given α and R, so that c_i, c_r and ψ_1 are known, equations (4.3) and (4.4) may be solved for ψ_2 and f. Thus for f we have

$$f = i\alpha R\int_1^z(\psi_1'\tilde{\psi}_1 - \tilde{\psi}_1'\psi_1)\,dz, \tag{4.8}$$

which is a real function of z. Knowing ψ_1, ψ_2 and f, we may then evaluate $g(z)$ and $g_1(z)$ from (4.5) and (4.6).

To solve equation (4.2), we look for a separable solution and write

$$g = kg_1(z) + h(z), \tag{4.9}$$
$$A_{11}(t) = A_1^2\tilde{A}_1, \tag{4.10}$$

where k is a complex number to be determined. [Another way of approach to this notion of separability is to look for a solution in which an equation of the form (4.12) is valid, since such an equation is related to the solution derived physically in the paper by Stuart (1958). The assumption (4.9), in which $g(z)$ has a part proportional to $g_1(z)$, then follows automatically.]

We then have

$$\left(ic_i A_1 - \frac{i}{\alpha}\frac{dA_1}{dt} - kA_1^2\tilde{A}_1\right) g_1(z)$$
$$+ A_1^2\tilde{A}_1\left[(\bar{u}_l - c_r)(\psi_{11}'' - \alpha^2\psi_{11}) - \bar{u}_l''\psi_{11} + \frac{i}{\alpha R}(\psi_{11}^{\mathrm{IV}} - 2\alpha^2\psi_{11}'' + \alpha^4\psi_{11}) - h(z)\right] = 0. \tag{4.11}$$

Equation (4.11) will be solved if the following two ordinary differential equations are soluble subject to the boundary conditions

$$\frac{dA_1}{dt} = \alpha c_i A_1 + i\alpha k A_1^2\tilde{A}_1, \tag{4.12}$$

$$(\bar{u}_l - c_r)(\psi_{11}'' - \alpha^2\psi_{11}) - \bar{u}_l''\psi_{11} + \frac{i}{\alpha R}(\psi_{11}^{\mathrm{IV}} - 2\alpha^2\psi_{11}'' + \alpha^4\psi_{11}) = h(z) = g - kg_1. \tag{4.13}$$

Equation (4.12) and its physical implications will be discussed in the next section, but first it is necessary to show how equation (4.13), together with its boundary conditions (4.7), leads to the determination of k as well as $\psi_{11}(z)$.

An apparent difficulty in the solution of (4.13) is that because c_i is very small, the form of the left-hand side of (4.13) differs only by a small term, of order c_i, from the form of (2.12). Consequently, one part of the complementary function of (4.13) differs from the normalized eigenfunction, $\psi_1(z)$, only by a term of order c_i. Since the boundary conditions on ψ_1 and ψ_{11} are the same, the problem of the determination of ψ_{11} is ill-conditioned. Watson (1960) has shown that the proper solution of this problem lies in considering the equation obtained by adding to (4.13) appropriate terms of order c_i, to yield

$$L\psi_{11} \equiv (\bar{u}_l - c_r - ic_i)(\psi_{11}'' - \alpha^2\psi_{11}) - \bar{u}_l''\psi_{11} + \frac{i}{\alpha R}(\psi_{11}^{\mathrm{IV}} - 2\alpha^2\psi_{11}'' + 2\alpha^4\psi_{11}) = g - kg_1. \tag{4.14}$$

The boundary conditions are

$$\left.\begin{aligned} \psi_{11} = \psi_{11}' = 0 \quad &\text{at} \quad z = 1, \\ \psi_{11}' = \psi_{11}''' = 0 \quad &\text{at} \quad z = 0. \end{aligned}\right\} \tag{4.15}$$

We may now determine k together with three constants of the complementary function, but *not* the constant multiplying the eigenfunction ψ_1.

In order to solve this problem most efficiently we need to consider the adjoint system to (2.12) and (2.13). The adjoint equation (cf. Ince 1956, pp. 210–14) is

$$\bar{L}\Phi \equiv (\bar{u}_l - c_r - ic_i)(\Phi'' - \alpha^2\Phi) + 2\bar{u}_l'\Phi' + \frac{i}{\alpha R}(\Phi^{\mathrm{IV}} - 2\alpha^2\Phi'' + \alpha^4\Phi) = 0, \tag{4.16}$$

and the adjoint boundary conditions (cf. Ince 1956, pp. 210–14) are

$$\Phi = \Phi' = 0 \quad \text{at} \quad z = 1, \qquad \Phi' = \Phi''' = 0 \quad \text{at} \quad z = 0. \tag{4.17}$$

For the case $\bar{u}_l \equiv 1 - z^2$, it can also be shown that, if the general solution of (2.12) is $\psi_1 \equiv \psi_{1g}$, then the general solution of (4.16) is $\Phi_g \equiv \psi_{1g}'' - \alpha^2\psi_{1g}$. Furthermore, (4.16) is the perturbation vorticity equation, though the conditions (4.17) are not the normal boundary conditions on vorticity.

If we multiply (4.14) by the solution Φ of (4.16) and (4.17), and integrate between 0 and 1, we may easily show that the left-hand side yields

$$\int_0^1 \Phi L\psi_{11}dz = \int_0^1 \psi_{11}\bar{L}\Phi\, dz. \tag{4.18}$$

This is identically zero by (4.16), so that the right-hand side gives

$$\int_0^1 \Phi(g-kg_1)\,dz = 0. \tag{4.19}$$

Thus

$$k = \frac{\int_0^1 \Phi g\,dz}{\int_0^1 \Phi g_1\,dz}. \tag{4.20}$$

Having determined k, we may obtain the solution of (4.13) and (4.14), except for the addition of an arbitrary multiple of the eigenfunction.

It should be borne in mind that complications arise if

$$\int_0^1 \Phi g_1 dz = 0. \tag{4.21}$$

In this unlikely case the present analysis is no longer valid, and it is necessary to use an expansion of the kind discussed by Watson (1960, equation (2.1.20)). This and other aspects of the problem receive detailed attention in Watson's paper.

It should be noted that the value of k given by the above analysis will, in general, have a finite value on the neutral curve ($c_i = 0$); moreover, as shown in § 5, the sign of the imaginary part of k is of great importance in determining the physical nature of the solution. The problem of the solution of (4.14) may be reformulated slightly, if desired, by expanding α, c_r, R and ψ_1 about a point on the neutral curve, in which case the number k is independent of c_i.

It is intended to carry out the calculations necessary for the determination of k.

5. The differential equation for disturbance growth

Consider equation (4.11) and its complex conjugate, namely

$$\frac{dA_1}{dt} = \alpha c_i A_1 + i\alpha k A_1^2 \tilde{A}_1, \tag{4.12}$$

$$\frac{d\tilde{A}_1}{dt} = \alpha c_i \tilde{A}_1 - i\alpha \tilde{k} \tilde{A}_1^2 A_1, \tag{4.12a}$$

where the parameter k may now be regarded as determined by the method described at the end of the last section. If we multiply (4.12) by $\tilde{A}_1$, (4.12a) by A_1 and add, we obtain

$$\frac{d\,|A_1|^2}{dt} = 2\alpha c_i\,|A_1|^2 - 2\alpha k_i\,|A_1|^4, \tag{5.1}$$

where k_i is the imaginary part of k. This equation has been discussed by Stuart (1958), with special reference to the case $\alpha c_i > 0$ (instability for small disturbances). The solution is

$$|A_1|^2 = \frac{c_i C \exp(2\alpha c_i t)}{1 + k_i C \exp(2\alpha c_i t)}, \tag{5.2}$$

where C is an arbitrary (real) constant.

If $c_i > 0$, which implies that the Reynolds number is greater than the critical value for the given values of α, it is clear that the solution (5.2) has a meaningful equilibrium value ($|A_1|^2 > 0$) if $k_i > 0$; in this case (figure 2) the disturbance amplifies in the way predicted by linearized theory ($|A_1|^2 \sim \exp(2\alpha c_i t)$) at $t = -\infty$, and tends to the equilibrium value $|A_1|_e^2 = c_i/k_i$ as $t \to +\infty$. If $k_i < 0$ there is no equilibrium amplitude, because $|A_1|^2$ must be positive. In either case, (5.2) gives a correct prediction only for such values of t that $|A_1|^2$ is bounded and of order c_i.

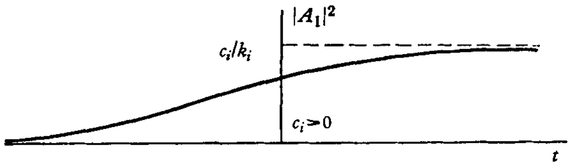

FIGURE 2. Growth of amplitude in supercritical case.

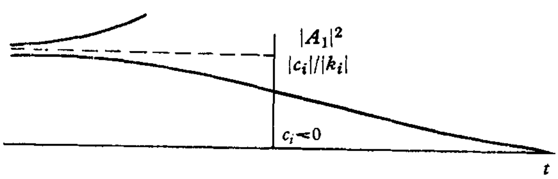

FIGURE 3. Growth of amplitude in subcritical case.

On the other hand, if $c_i < 0$ the solution (5.2) has a meaningful equilibrium value if $k_i < 0$; the disturbance (figure 3) takes on the equilibrium value $|A_1|_e^2 = |c_i|/|k_i|$ at $t = -\infty$, but the equilibrium is unstable. If the amplitude is slightly less than this equilibrium value, the disturbance decays to zero amplitude via the damped solution of linearized theory ($|A_1|_e^2 \sim \exp(-2\alpha|c_i|t)$). However, if the amplitude is slightly greater than the equilibrium value the disturbance grows in amplitude. Equation (5.2) does not indicate what happens to a disturbance which grows in this way, because according to (5.2) the amplitude tends to infinity and is not restricted to the amplitude range $|A_1|^2 \sim c_i$.

If we now multiply equation (4.12) by $1/\tilde{A}_1$, equation (4.12a) by $(-A_1/\tilde{A}_1^2)$ and add, we obtain

$$\frac{d}{dt}\left(\frac{A_1}{\tilde{A}_1}\right) = 2ik_r\alpha A_1^2, \tag{5.3}$$

where k_r is the real part of k. Equation (5.3) may be solved to yield

$$\frac{A_1}{\tilde{A}_1} = \exp\left\{2i\alpha k_r \int_{t_0}^{t} |A_1|^2\, dt\right\}, \tag{5.4}$$

where t_0 is an arbitrary (real) constant. By multiplying (5.4) by $|A_1|^2$ and taking the square root, we obtain

$$A_1 = |A_1| \exp\left\{i\alpha k_r \int_{t_0}^{t} |A_1|^2\, dt\right\}, \tag{5.5}$$

where $|A_1|$ is given by (5.2).

In interpreting (5.5), we consider first the case $c_i > 0$, $k_i > 0$. When $t \to -\infty$,

$$A_1 \to |A_1|\, e^{i\gamma} = (c_i C)^{\frac{1}{2}} \exp(\alpha c_i t + i\gamma), \tag{5.6}$$

where γ is a number, independent of t but dependent on t_0. On the other hand, when $t \to +\infty$,

$$A_1 \to \left(\frac{c_i}{k_i}\right)^{\frac{1}{2}} \exp\left\{\frac{i\alpha k_r c_i}{k_i}(t-t_k')\right\}, \tag{5.7}$$

where t_k' is an arbitrary phase (which is related to the arbitrary number t_0). The importance of formulae (5.6) and (5.7) lies in the fact that they show that, if A_i is non-oscillatory at $t = -\infty$, it develops a fluctuation as t progresses and has the limiting form (5.7) as $t \to +\infty$. Using (2.4) and (4.1), we see that (to order $c_i^{\frac{1}{2}}$)

$$\phi_1(z,t)\exp[i\alpha(x-c_r t)] = A_1(t)\,\psi_1(z)\exp[i\alpha(x-c_r t)] \tag{5.8}$$

has the limiting forms

$$c_i^{\frac{1}{2}}(C^{\frac{1}{2}}e^{i\gamma})\,\psi_1\exp[i\alpha(x-c_r t-ic_i t)] \quad \text{as} \quad t \to -\infty, \tag{5.9}$$

and

$$\left(\frac{c_i}{k_i}\right)^{\frac{1}{2}}\psi_1(z)\exp\left\{i\alpha\left[x-\left(c_r-\frac{k_r c_i}{k_i}\right)(t-t_k)\right]\right\} \quad \text{as} \quad t \to +\infty. \tag{5.10}$$

Formula (5.9) is merely the amplifying disturbance of linearized theory, with $C^{\frac{1}{2}}e^{i\gamma}$ the arbitrary amplitude factor. Formula (5.10) shows that in the ultimate equilibrium state the wave velocity is

$$c_{re} = c_r - \frac{k_r c_i}{k_i}. \tag{5.11}$$

It will also be noticed that there is an arbitrary phase (t_k) in the fluctuation.

For the case $c_i < 0$, $k_i < 0$, we can obtain similar results to those given above; we find, in fact, that (5.8) has the limiting forms

$$(c_i C)^{\frac{1}{2}}e^{i\gamma}\,\psi_1(z)\exp[i\alpha(x-c_r t-ic_i t)] \quad \text{as} \quad t \to +\infty, \tag{5.12}$$

and

$$\left(\frac{|c_i|}{|k_i|}\right)^{\frac{1}{2}}\psi_1(z)\exp\left\{i\alpha\left[x-\left(c_r-\frac{k_r|c_i|}{|k_i|}\right)(t-t_k)\right]\right\} \quad \text{as} \quad t \to -\infty. \tag{5.13}$$

Equation (5.12) is the damped linearized solution, with $(c_i C)^{\frac{1}{2}}e^{i\gamma}$ as the arbitrary amplitude factor. Formula (5.13) shows that, in the (unstable) equilibrium state at $t = -\infty$, the wave velocity is

$$c_{re} = c_r - \frac{k_r|c_i|}{|k_i|}. \tag{5.14}$$

It should be appreciated that, in the present paper, only a limiting non-linear solution (as $c_i \to 0$) has been obtained; no attempt has been made to develop a perturbation series. This, however, has been done by Watson (1960), who has shown that equation (4.12) has to be replaced by a generalized version, namely

$$\frac{dA_1}{dt} = A_1(\alpha c_i + a_1|A_1|^2 + a_2|A_1|^4 + \ldots),$$

where a_1, a_2 are constants to be determined in a way similar to that in which it is suggested that k be determined in the present paper.

6. Discussion

As described at the end of § 3, a condition for the validity of the solution described in this paper and in that of Watson (1960), is that the parameter

$\vartheta = c_i(\alpha R)^{\frac{1}{3}}$ must be sufficiently small compared with unity. Although the solution has been derived and explained in this paper on the basis of $c_i \to 0$, it should be recalled that this condition has to be interpreted in terms of the parameter ϑ; in fact the expression $c_i \to 0$ means 'for sufficiently small values of ϑ'. We can evaluate ϑ for various wave-numbers and Reynolds numbers by reference to the calculations of Thomas (1953) and Shen (1954). The minimum critical Reynolds number occurs at $R = 5780$ for $\alpha = 1{\cdot}02$. Calculation shows that ϑ is about $0{\cdot}08$ at $\alpha = 1$, $R = 10^4$; $\vartheta = 0{\cdot}269$ at $\alpha = 0{\cdot}77$, $R = 5{\cdot}6 \times 10^4$; $\vartheta = 0{\cdot}32$ at $\alpha = 0{\cdot}7$, $R = 12{\cdot}5 \times 10^4$. These values of ϑ are approximately the largest at the given values of R. From these and other results it seems that Watson's series will probably converge over the whole wave-number band within the neutral curve for a wide range of Reynolds number, certainly up to $R = 10^4$ and possibly to $R = 12{\cdot}5 \times 10^4$. The method of the present paper is also likely to be a good approximation within the neutral curve up to $R = 10^4$ (which is about twice the minimum critical Reynolds number), and possibly even to higher Reynolds numbers. Thus, at a given Reynolds number in the supercritical range from $R = 5780$ to $R = 10^4$ or 10^5, we may expect the method of solution described in this paper and in Watson's to be a valid means of calculating equilibrium states, if they exist, for the whole band of unstable wave-numbers.

At lower Reynolds numbers, where the flow is stable according to linearized theory. Thomas's calculations yield

$$\vartheta = -0{\cdot}152 \quad \text{at} \quad R = 2500, \quad \alpha = 1{\cdot}1;$$
$$\vartheta = -0{\cdot}248 \quad \text{at} \quad R = 1600, \quad \alpha = 1{\cdot}1.$$

Thus one would expect to be able to calculate subcritical equilibrium states, if they exits, for a large range of Reynolds numbers (possibly down to $R = 2500$ and below); but the band of wave-numbers is relatively narrow.

The approximate calculation of Meksyn & Stuart (1951) led to the evaluation of subcritical equilibrium states in the range of Reynolds number down to about 3000. In that paper the movement of the 'critical' point ($\bar{u} = c$) was of order one-eighth of the thickness of the critical layer. Consequently, the present method can be expected to show the validity or otherwise of the method used in the earlier paper.

The differential equation (5.1) for the square of the modulus of the disturbance amplitude can be shown to be an energy-balance relation for the fundamental (ϕ_1) disturbance, in which the rate of increase of disturbance energy equals the net flow of energy to the fundamental less the viscous dissipation of energy. If we define u', w' to represent the velocity components of that part of the disturbance which has odd wave-number (α, 3α, etc. of (2.4)), and u'', w'' to represent the velocity components of that part of the disturbance which has even wave-numbers (2α, 4α, etc.), then it can be shown that

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2}(u'^2+w'^2)\,dx\,dz = \iint(-u'w')\frac{\partial\bar{u}}{\partial z}\,dx\,dz - \frac{1}{R}\iint\left(\frac{\partial w'}{\partial x}-\frac{\partial u'}{\partial z}\right)^2 dx\,dz$$
$$-\iint\left[(u'^2-w'^2)\frac{\partial u''}{\partial x}+u'w'\left(\frac{\partial u''}{\partial z}+\frac{\partial w''}{\partial x}\right)\right]dx\,dz, \quad (6.1)$$

where the integration ranges over one wavelength $(2\pi/\alpha)$ and between the planes. This equation states that the rate of increase of energy in the 'odd' part of the disturbance (u', w') equals the rate of transfer of energy from the mean motion, less the rate of dissipation of energy, less the rate of transfer of energy from the 'odd' to the 'even' (u'', w'') part of the disturbance. To the order of approximation of this paper u', w' are given by the stream function ϕ_1 of (4.1), and u'', w'' by the stream function ϕ_2; thus (6.1) is an equation for rate of change of energy of the fundamental.

By substituting (2.4) and (4.1) into (6.1), with appropriate definitions of the velocities in terms of the stream functions, we can obtain

$$\frac{d\,|A_1|^2}{dt} = 2\alpha c_i\,|A_1|^2 + (k_1+k_2+k_3)\,|A_1|^4, \tag{6.2}$$

where some terms of order c_i in the coefficient of $|A_1|^4$ have been ignored (together with higher power of $|A_1|^2$ in Watson's expansion), and

$$k_1 = \frac{-i\alpha}{k_0}\int_0^1 (\psi_1'\tilde{\psi}_1 - \psi_1\tilde{\psi}_1')f'\,dz, \tag{6.3}$$

$$k_2 = \frac{-i\alpha}{k_0}\int_0^1 (2\psi_2'\tilde{\psi}_1'^2 + \psi_2''\tilde{\psi}_1'\tilde{\psi}_1 - 2\tilde{\psi}_2'\psi_1'^2 - \tilde{\psi}_2''\psi_1'\psi_1)\,dz, \tag{6.4}$$

$$k_3 = \frac{2i\alpha}{k_0}\int_0^1 (\psi_1'\tilde{\psi}_{11} + \psi_{11}'\tilde{\psi}_1 - \tilde{\psi}_1'\psi_{11} - \tilde{\psi}_{11}'\psi_1)\,z\,dz$$
$$-\frac{2}{k_0R}\int_0^1 [(\psi_1'' - \alpha^2\psi_1)(\tilde{\psi}_{11}'' - \alpha^2\tilde{\psi}_{11}) + (\tilde{\psi}_1'' - \alpha^2\tilde{\psi}_1)(\psi_{11}'' - \alpha^2\psi_{11})]\,dz, \tag{6.5}$$

$$k_0 = \int_0^1 [|\psi_1'|^2 + \alpha^2\,|\psi_1|^2]\,dz. \tag{6.6}$$

Comparison with equation (5.1) suggests that

$$-2\alpha k_i = k_1 + k_2 + k_3, \tag{6.7}$$

and in fact this relation can be deduced mathematically from the equations of § 4.

The three parts of the coefficient of the fourth power of (6.2) arise from the following physical processes:

(i) the distortion of the mean motion (k_1);

(ii) the generation of the harmonic of the fundamental (k_2);

(iii) distortion of the fundamental, with regard to its dependence on z (k_3).

It is instructive to consider the signs of the coefficients k_1, k_2 and k_3. Substitution of (4.8) into (6.3) shows that k_1 is negative, this term being exactly the one calculated as the coefficient of $|A_1|^4$ in the energy-balance method (Stuart 1958). (In the derivation of (4.8) terms of order c_i were omitted, so that k_1 has an error of order c_i; this is negligible to the order we consider, since terms of order c_i^3 in (6.2) are omitted.) It is now seen that if k_1 alone is retained, as in the energy-balance method, we must obtain a supercritical equilibrium state.

The estimation of the signs of k_2 and k_3 is more difficult, and it does not appear to be possible to do more than speculate. It seems likely that k_2 will be negative; it represents flow of energy from the fundamental to the first harmonic, which is maintained only by this energy flow. The coefficient k_3 represents the modification to the net energy flow into the fundamental, due to z-distortion of the fundamental; the author knows of no argument which would suggest the sign of k_3. (It may be noted that the paper of Meksyn & Stuart (1951) included processes (i) and (iii) approximately but excluded process (ii).)

The net effect of the three processes described above can only be obtained by completion of the calculations described in this paper. However, it is possible to say that, if a subcritical equilibrium state is to be obtained, k_3 must be positive and must outweigh the combined negative effect of k_1 and k_2. It is felt that a major contribution of this paper and of Part 2 is that all the physical processes described above are included in mathematical form.

The extension of the above analysis to include three-dimensional effects has been described elsewhere (Stuart 1960).

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